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Calculus via limit sets

For a sequence f of real numbers $f(n)$, $n = 1, 2, ...$, write $f(n+1) = f(n) + f'(n)$, with $f'(n) := f(n+1) - f(n)$ and $f(1)$ prescribed. Executing the iteration gives $f(n+1) = f(1) + f'(1) + ... + f'(n)$, a representation of f as the "integral" of its "derivative" f'. Rearrange terms in the sum, and $f(n+1)$ becomes the difference of $f(1) + f(2) + ... + f(n+1)$ and $f(1) + f(2) + ... + f(n)$, a representation of f as the "derivative" of its "integral". These trivial observations, and the classical notion of limit of a function, yield classical calculus; it extends deterministic models of computation. Suppose now f is a sequence of sets of real numbers. It still makes sense to form the "derivative" and "integral" of f, both of which are now set-valued, their relationship carrying over with obvious modifications. This, and the notion of limit of a set-valued map, yield set-valued calculus; it extends non-deterministic models of computation.

Set-valued analysis is a well-developed subject, but, to my knowledge, the calculus of set-valued maps in the reals still awaits orthodox formulation. Set-valued calculus might compete on didactic grounds with strict expositions of standard calculus, but hardly with the intuitive calculus syllabi common today. But there is a compromise: to employ set-valued maps to analyze functions. This indeed simplifies the logic of calculus considerably due to the local com-pactness of the reals, an elementary fact bypassed by elementary expositions. All here is "visual": limits of a function in a compactified Euclidean space are read off from the closure of its graph.

We thus change little. The derivative and the integral are taken in the usual way, via difference quotient and Riemann sum, though not as point limit but as closure of graph; the definitions are meaningful for set-valued maps. For the Riemann integral this approach is particularly natural, the standard Riemann sum in effect always having been set-valued. We close maps before analysis; we let the derivative of a map, for example, be the derivative of its closure. For an arbitrary set-valued map, the derivative and the integral exist as set- valued maps, reducing to classical versions if point-valued. Standard theorems of calculus get simple form, and standard classes of functions get simple characterization. For example, Lipschitz functions are maps with bounded derivative, monotone functions are functions with derivative of constant sign, and convex functions are maps with positive second derivative.

Key words: *elementary calculus*, *set-valued map*, *closure of graph*

[1] R. F. Bonner, *Elementary analysis via closure of graph*, Report 2004-1, Dept. of Mathematics and Physics, Mälardalen University, Västerås, Sweden, 2004.