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Hausdorff moment problem and polylogarithm

The Hausdorff moment problem is to find a Borel probablity measure μ on the interval $[0, 1]$ such that

$$\int_0^1 t^n d\mu(t) = a_n, \quad n = 0, 1, 2, \dots,$$

for a given sequence $a_0, a_1, \dots$ of positive numbers. It is known that the problem is (uniquely) solvable if and only if the sequence is totally monotone, that is, $\Delta^k a_n \geqslant 0$ for every $k \geqslant 1$ and $n \geqslant 0$. Here,

$$\Delta^k a_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n+j}.$$

The analytic function f given as the power series $f(z) = a_0 + a_1 z + a_2 z^2 + \dots$ is then represented by

$$f(z) = \int_0^1 \frac{d\mu(t)}{1 - tz}.$$

In particular, f satisfies $\Re f(z) > 1/2$ in $|z| < 1$ and f is analytically continued in the slit domain $\mathbb{C} \setminus [1, +\infty)$ and $\Im f(z) > 0$ for $\Im z > 0$.

It is known that the sequence $a_n = (n + 1)^{-\alpha}$ is totally monotone for $\alpha \geqslant 0$. Thus the polylogarithm $f_\alpha(z) = \sum_{n=1}^{\infty} z^n/n^\alpha$ can be represented in the form $f_\alpha(z) = z \int_0^1 d\mu(t)/(1 - tz)$. St. Ruscheweyh conjectured that $1 + z f_\alpha''(z)/f_\alpha'(z)$ is also generated by a totally monotone sequence for $\alpha \geqslant 1$. We prove this conjecture. In particular, we see that f_α is convex of order $1/2$ for $\alpha \geqslant 1$. Note that C. J. Lewis [1983, J. LMS] showed that f_α is convex if and only if $\alpha \geqslant 0$.